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Theory of Games.

Introduction: Life is a full of struggle and Competitions. A great variety of competitive situations is commonly seen in everyday life. For example, Candidates fighting an election have their conflicting interests, because each candidate is interested to secure more votes than those secured by all others. Besides, we come across much more earnest competitive situations of advertising, marketing Campaigns by competing business firms etc. Game must be thought of, in a broad sense, not as a kind of sport but as competitive situation, a kind of conflict in which somebody must win and somebody must loss.

Game Theory is a type of decision theory in which one's choice of action is determined after taking into account all possible alternatives available to an opponent playing the same game, rather than just by the possibilities of several outcomes. The mathematical analysis of competitive problems is fundamentally based upon the minimax (maximin) Criterion of J. Von Neumann (called the father of Game theory). This Criterian implies the assumption of rationality from which it is ~~ay~~ argued that each player will act so as to maximize his minimum gain or minimize his maximum loss. The difficulty lies in the deduction from the assumption of 'rationality' that the other player will maximize his minimum gain. There is no agreement even among game theorists that rational players should so act. In fact, rational players do not act apparently in this way, or in any consistent way. Therefore, game theory is generally interpreted as an "as if" theory, that is, as if rational decision maker (player) behaved in some well defined way, such as maximizing the minimum gain.

The game theory has only been capable of analysing very simple competitive situations.

Game is defined as an activity between two or more persons involving activities by each person according to a set of rules, at the end of which each person receives some benifits or satisfaction or suffers loss (negative benefit).

The set of rules defines the game. Going through the set of rules once by the participants defines a play.

Let us consider an example of game.

Let A and B be two business men working in the same field (ignore the presence of the 3rd party in the field). There must be some conflicts of interest between them regarding business matters. In the mathematical terminology the business men A and B are called players and the business is called as game.

Suppose A has three executives A_1, A_2 and A_3 and B has four executives B_1, B_2, B_3, B_4 to control their respective business. We make a restriction that both players ~~will~~ utilize the services of their executives only one at a time to control their whole business.

For example, A may utilize the service of A_2 and B may utilize the service of B_3 only to control their business. But the selection of their executives entirely depends upon their own discretion ignoring the selection, done by the other players.

The selection of a particular executive (or course of action) is called the strategy. The selection of only one single move or strategy by a player, ignoring the strategy taken by ~~the other~~ his opponent is called a pure strategy.

First of all, we consider the games relating to the pure strategy taken by the players.

We make two assumptions

(i) The player A is in a better position ie the player A is called the maximising player (row player) & the player B is called the minimizing player (column player)

(ii) The total gain of one player is exactly equal to the total loss of the other player ie the sum total is zero.

Thus the game is referred as a conflict situation and competition between two opponents. Thus the game theory deals with making decisions under conflict caused by opposing interests.

<u>Pay-off-matrix</u>: A table showing how payments should be made at the end of the game is called a pay-off-matrix.

Otherwise we state that

Thus ~~A~~ pay-off-matrix is a real matrix (a_{ij}), $i = 1, \cdots, m$, $j = 1, 2, \cdots, n$ (in general a rectangular matrix) which are the elements a_{ij} indicates the gain of the maximizing player for using the ith and jth move of the row and column players respectively.

<u>Zero-Sum Game</u>: A game in which the gain ~~the~~ of one player is a loss to another is called a zero-sum game.

Two person zero-sum game / rectangular game:
If the losses of one player be equivalent to the gains of another player, then the sum of the gains to the players being zero after the game, then it is called a two-person zero-sum game or a rectangular game.

Strategy: A strategy as defined earlier is a set of pre-determined rules or programmes by which a player will decide which of the available courses of action he will have to adopt at each play.

The game problem may represent a deterministic situation in which the object is to maximize the gain, or it may represent some probabilistic situation in which the object of the maximizing player will be to maximize his expected gain.

In the former each player knows in advance the courses of action that may be taken by the other player and he will then chose a particular course and in the latter a player is kept guessing as to the possible course of action to be adopted by the opponent.

Pure ~~str~~ strategy: A pure strategy is a decision making rule in which one particular course of action is selected, ignoring the strategy taken by the other.

Mixed strategy: A mixed strategy is a decision making rule in which a player decides, in advance, to choose his courses of with some definite probability. In the mixed strategy, one player is always kept guessing about the opponent's choice. A player has a finite no. of pure strategy but has infinite no. of mixed strategy.

In a mixed strategy, selection is made of all pure strategy with some definite probabilities.

If a player has m pure strategies, then the set of real numbers $\{x_1, x_2, \cdots, x_m\}$ (probability x_j corresponds to the jth strategy) will be a mixed strategy for him if

$$x_1 + x_2 + \cdots + x_m = 1$$
$$x_j \geq 0, \; j = 1, 2, \cdots, m.$$

Fair game: If the value of the game zero, then we call the game a fair game.

Maximin minimax principle:

Step 1: Calculate the value of the row minimum for each row and write down the m values in a column under the heading "Row minima".

Find out the maximum value of the elements in that column headed by "Row minima". Let it be α and let it occur at kth row.

Step 2: Calculate the value of the column maximum for each column and write down the n values in a row under the heading 'Column maxima'.

Find the minimum value of the elements of the row under the heading "Column maxima".

Let it be β and let it occur at l-th column.

<u>Step 3</u>: In general $\beta \geq \alpha$ and if $\beta = \alpha$, the value of the game $\alpha = \beta$ and the optimal strategies are A_k and B_l for A & B respectively, and the <u>saddle point</u> of the pay-off matrix is (k, l) th position of the matrix.

<u>Note</u>: If the pay-off matrix be $[a_{ij}]_{m \times n}$ then 'maximum of the minimum gains' for A is mathematically expressed as $\max_i \min_j a_{ij}$ and "minimum of the maximum losses" for B is denoted by

$$\min_j \max_i a_{ij} .$$

Again, $\max_i \min_j a_{ij} = \underline{v}$

& $\min_j \max_i a_{ij} = \overline{v}$

We have in general $\underline{v} \leq v \leq \overline{v}$.

$\therefore$ Value of the game = max (row min) = min (col. max)

If a matrix involves more than one saddle point, then there exist more than one solution of the game.

<u>Ex-1</u>. Solve the following game whose payoff matrix is given by:

$$\begin{array}{cc} & B \\ & \begin{array}{cccc} I & II & III & IV \end{array} \\ A \begin{array}{c} I \\ II \\ III \end{array} & \left[\begin{array}{cccc} -5 & 3 & 1 & 20 \\ 5 & 5 & 4 & 6 \\ -4 & -2 & 0 & -5 \end{array}\right] \end{array}$$

<u>Ans</u>: Let A be the maximizing player and B be the

minimizing player.

We find out the row minimum in a column and column maximum in a row. Thus the pay-off matrix is given below.

$$\begin{array}{cc|cccc|c} & & & & B & & \\ & & \mathrm{I} & \mathrm{II} & \mathrm{III} & \mathrm{IV} & \text{Row min} \\ \hline & \mathrm{I} & -5 & 3 & 1 & 20 & -5 \\ A & \mathrm{II} & 5 & 5 & 4 & 6 & 4 \\ & \mathrm{III} & -4 & -2 & 0 & -5 & -5 \\ \hline & \text{Column max} & 5 & 5 & 4 & 20 & \end{array}$$

Again find out the max (row min) and min (col. max).

Here max (row min) = 4 = min (Col. max).

Now it is seen that the element 4 of the 2nd row and 3rd column has been marked around both by circle. Then this position is the saddle point of the pay-off-matrix. and the pure strategy II for A and III for B are the optimal solution of the game which is written as (II, III),

Thus 4 is the value of the game as

$$\underline{v} = 4 = \overline{v}.$$

Since $\underline{v} = \text{Max}\{-5, 4, -5\} = 4$ & $\overline{v} = \text{Min}\{5, 5, 4, 20\}$
$= 4$.

Ex 2 : Show that whatever may be the value of a, the game with the following pay-off-matrix is strictly determinable:

$$\begin{array}{cc|cc|} & & B & \\ & & \mathrm{I} & \mathrm{II} \\ \hline & \mathrm{I} & 3 & 7 \\ A & \mathrm{II} & -3 & a \\ \hline \end{array}$$

Ans : We 1st ignore a. The given pay-off matrix (ignoring a) gives the following row minimum &

Column maximums

$$\begin{array}{cc|cc|c} & & \multicolumn{2}{c|}{B} & \\ & & I & II & \text{Row min} \\ A & I & 3 & 7 & 3 \\ & II & -3 & a & -3 \\ \hline & \text{Col max} & 3 & 7 & \end{array}$$

$\therefore$ $\underline{v} = 3$ & $\bar{v} = 3$ whatever may be the value of a.

$\therefore$ The game is strictly determinable & $v = 3$.

The best strategy for A & B are (I, I), whatever may be the value of a.

Ex-3 : For what value of a, the game with the following pay-off-matrix is strictly determinable?

$$\begin{array}{cc|ccc|} & & \multicolumn{3}{c}{B} \\ & & I & II & III \\ & I & a & 5 & 2 \\ A & II & -1 & a & -8 \\ & III & -2 & 3 & a \end{array}$$

Ans : We first ignore a and determine $\bar{v}$ & $\underline{v}$ of the pay-off-matrix by computing the row minimum and the column maximums.

For this, we have the matrix below.

$$\begin{array}{cc|ccc|c} & & \multicolumn{3}{c|}{B} & \\ & & I & II & III & \text{Row min} \\ & I & a & 5 & 2 & 2 \\ A & II & -1 & a & -8 & -8 \\ & III & -2 & 3 & a & -2 \\ \hline & \text{Col. max} & -1 & 5 & 2 & \end{array}$$

We have maximin for A = 2 & minimax for B = −1.

$\therefore \underline{v} = -1, \quad \overline{v} = 2.$

The strategy for A is I and for B it is I for these values.

If the game is to determinable with its value v, then v should be such that $-1 \le v \le 2$.

Thus, for a strictly determinable game,

$$-1 \le a \le 2.$$

Ex-4: Show that the 2×2 game $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is non-strictly determinable if $a<b$, $a<c$, $d<b$ & $d<c$.

Ans: Since $a<b$ & $d<c$, therefore the row minimums for the two rows are respectively a & d.

Now $\underline{v} = \max(\text{row min}) = \max(a,d)$. —— ①

Again, since, $a<c$, $d<b$, therefore the column maximums of the two columns are respectively c and b.

Now $\overline{v} = \min(\text{column max})$

$= \min(c,b)$ —— ②

From ① & ② we show that $\overline{v} \neq \underline{v}$

Hence the game is non-strictly determined.

Ex-5: Find the range of values of p & q which will render the entry (2,2) is a saddle point for the game

$$\begin{array}{cc|ccc|} & & \text{I} & \text{II} & \text{III} \\ & & & B & \\ \hline & \text{I} & 2 & 4 & 5 \\ A & \text{II} & 10 & 7 & q \\ & \text{III} & 4 & p & 6 \\ \hline \end{array}$$

Ans: First ignoring the values of p & q determine the maximin and minimax values of the pay-off matrix as below.

Since the entry $(2,2)$ is a saddle point, maximin $\underline{v} = 7$, minimax value $\bar{v} = 7$.

This imposes the cond^n on p as $p \le 7$ and on q as $q \ge 7$. Hence the range of p & q will be

$$p \le 7, \quad q \ge 7.$$

Note: [The maximin value of the game, called the lower value, is denoted by $\underline{v}$ and the minimax value, called the upper value is denoted by $\bar{v}$.

Thus for a strictly determinable game

$$\underline{v} = v = \bar{v}$$

and for a fair game, $\underline{v} = 0 = \bar{v}$.

The game is said to be non-strictly determined if $\bar{v} \neq \underline{v}$

If $v \neq 0$, then the game is called the biased. If $v > 0$ then it is said to biased to the maximizing player and if $v < 0$, then it is said to biased to the minimizing player.]

Ex-5 Player A can choose his strategies from $\{A_1, A_2, A_3\}$ only, while player B can choose from the set $\{B_1, B_2\}$ only. The rules of the game state that payments should be made in accordance

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with the selection of strategies.

Strategy selected	Payments to be made
(A_1, B_1)	A pays Rs. 1 to B
(A_1, B_2)	B pays Rs 6 to A
(A_2, B_1)	B pays Rs. 2 to A
(A_2, B_2)	B pays Rs. 4 to A
(A_3, B_1)	A pays Rs 2 to B
(A_3, B_2)	A pays Rs 6 to B

What strategies should the players taken in the above game in order to get the maximum benefit?

Ans: Here A player can choose three strategies A_1, A_2 & A_3 and also B player chooses two strategies B_1, B_2.

Now we form a pay-off matrix for the player A and B.

B player

A player	B_1	B_2
A_1	-1	6
A_2	2	4
A_3	-2	-6

Let A be the maximing player and B be the minimizing player (Column player). We find out the row minimum in a column and column maximum in a row. Thus the pay-off matrix is given below

B

A	B_1	B_2	Row min
A_1	-1	6	-1
A_2	2	4	2
A_3	-2	-6	-6
Col. max	2	6	

* Using the maximin minmax principle.

$$\underline{v} = \max(\text{row min}) = 2$$

$$\bar{v} = \min(\text{col. max}) = 2.$$

$$\therefore \bar{v} = v = \underline{v} = 2.$$

Thus the value of the game is 2.

The strategy for A is A_2 and for B is B_1.

Solution of 2×2 Games (Mixed Strategy)

Consider the game whose payoff is given by the adjacent matrix and for which there is no saddle point. We are to determine the probabilities with which each strategy of A and B will be played to get an optimal solution.

Let x_i & y_j be the probabilities with which A chooses his ith strategy and B chooses his jth strategy respectively.

$$A \quad \begin{array}{c|cc} & \multicolumn{2}{c}{B} \\ & C_1 & C_2 \\ \hline R_1 & a_{11} & a_{12} \\ R_2 & a_{21} & a_{22} \end{array}$$

Then for this problem, the mixed strategies of A and B are respectively, $X = (x_1, x_2)$, $Y = (y_1, y_2)$ such that

$$x_1 + x_2 = 1$$
$$y_1 + y_2 = 1 \quad \cdots\cdots \quad (I)$$
$$x_1, x_2 \geq 0, \; y_1, y_2 \geq 0$$

The expected gain for A when B chooses C_1 or C_2 respectively $a_{11}x_1 + a_{21}x_2$ and $a_{12}x_1 + a_{22}x_2$

Similarly the expected loss to B for the solutions R_1 & R_2 by A respectively $a_{11}y_1 + a_{12}y_2$ and $a_{21}y_1 + a_{22}y_2$.

(7) Assuming that each rectangular game has a solution & assuming v to be the expected value of the game considered, we should have

$$\left.\begin{array}{l} a_{11}x_1 + a_{21}x_2 \geq v \\ a_{12}x_1 + a_{22}x_2 \geq v \end{array}\right\} \longrightarrow (2)$$

as A expects to get at least v.

Similarly, B expects to loss at most v & hence

$$\left.\begin{array}{l} a_{11}y_1 + a_{12}y_2 \leq v. \\ a_{21}y_1 + a_{22}y_2 \leq v. \end{array}\right\} \longrightarrow (3)$$

Assuming the above inequations as strict equations and assuming the existance of their solutions, we can say that the solutions (x_1, x_2) & (y_1, y_2) are the optimal solutions for the game.

Then (2) & (3) considered togather with equations (1) will x_1, x_2, y_1, y_2. Now if these satisfy the non-negativity restrictions given in (1), then this will be the optimal solution, otherwise we shall have to take recourse to algebric method for the solution of the game problem.

Considering the inequations (2) as strict equations, we get

$$a_{11}x_1 + a_{21}x_2 = a_{12}x_1 + a_{22}x_2$$

$$\Rightarrow \frac{x_1}{x_2} = \frac{a_{22} - a_{21}}{a_{11} - a_{12}} \longrightarrow (4)$$

Similarly from the inequations (3) we have

$$\frac{y_1}{y_2} = \frac{a_{22} - a_{12}}{a_{11} - a_{21}} \quad (5)$$

R2

From (4) & (1) we get

$$x_1 = \frac{a_{22} - a_{21}}{a_{11} + a_{22} - (a_{12} + a_{21})}$$

$$x_2 = \frac{a_{11} - a_{12}}{a_{11} + a_{22} - (a_{12} + a_{21})}.$$

Similarly, from equation (5) & the 2nd equation of (1), we have

$$y_1 = \frac{a_{22} - a_{12}}{a_{11} + a_{22} - (a_{12} + a_{21})}$$

$$y_2 = \frac{a_{11} - a_{21}}{a_{11} + a_{22} - (a_{12} + a_{21})}.$$

Then the expected value v of the game can be computed as

$$v = \frac{a_{11}a_{22} - a_{12}a_{21}}{a_{11} + a_{22} - (a_{12} + a_{21})}$$

Note:

Definition: Pay off function: Let $[a_{ij}]$ be any payoff matrix of order $m \times n$. Then the pay off function or mathematical expectation of a game which is denoted by $E(X,Y)$ is defined as

$$E(X,Y) = \sum_{i=1}^{m}\sum_{j=1}^{n} a_{ij}\, x_i x_j \quad \text{where } X = \{x_1, x_2, \dots x_m\}$$
$$Y = \{y_1, y_2, \dots, y_n\}.$$

In particular, if B takes his pure jth move only then the expected gain of A is

$$E_j(X) = \sum_{i=1}^{m} a_{ij} x_i, \quad j = 1, 2, \dots, n$$

Similarly, for particular ith move of A only, the expected loss of B is given by

$$E_i(Y) = \sum_{j=1}^{n} a_{ij}\, y_j, \quad i = 1, 2, \dots, m.$$

Ex

Find the value of 2×2 game algebraically by using mixed strategies

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(8)

$$A \quad \begin{array}{c c} & \begin{array}{cc} B_1 & B_2 \end{array} \\ \begin{array}{c} A_1 \\ A_2 \end{array} & \begin{bmatrix} 2 & 3 \\ 4 & -1 \end{bmatrix} \end{array}$$

we

Ans: The problem has no saddle point in the case of pure strategy in the pay off matrix for A.

Let us try to solve the problem by using mixed strategies $X = (x_1, x_2)$ and $Y = (y_1, y_2)$ with $x_1 + x_2 = 1$, $y_1 + y_2 = 1$, $x_1, x_2, y_1, y_2 \geq 0$ for A & B respectively.

Assuming the expected value of game for A when B_1 & B_2 are fixed,

$$E_1(X) = 2x_1 + 4x_2$$
$$E_2(X) = 3x_1 - x_2$$

To determine the optimal ~~strategy~~ values of x_1 & x_2 we have

$$2x_1 + 4x_2 = v = 3x_1 - x_2$$
$$\Rightarrow 2x_1 + 4x_2 = 3x_1 - x_2$$
$$\Rightarrow 2x_1 + 4(1-x_1) = 3x_1 - (1-x_1)$$
$$\Rightarrow x_1^* = 5/6 > 0 \quad \& \quad x_2^* = 1 - x_1^* = 1/6.$$

and the value of the game is

$$v = 2x_1^* + 4(1 - x_1^*)$$
$$= 7/3$$

Again considering from the B's point of view, we have

$$E_1(Y) = 2y_1 + 3y_2$$
$$E_2(Y) = 4y_1 + (-y_2).$$

To determine the optimal solⁿ of y_1 & y_2 we have

$$2y_1 + 3(1-y_1) = 4y_1 - (1-y_1)$$
$$\Rightarrow y_1^* = 2/3 > 0, \quad y_2^* = 1 - y_1^* = \frac{1}{3}$$

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& the value of the game is

$$2y_1^* + 3y_2^* = 2 \times \frac{2}{3} + 3 \times \frac{1}{3} = 7/3$$

Hence the optimal strategy are

$$X^* = (5/6, 1/6),\ Y^* = (2/3, 1/3)$$
$$\& \ v = 7/3$$

Solution of rectangular games with mixed strategy. (without saddle point).

In a game problem in which the pay-off matrix has no saddle point, the players are to play their strategies according to some probability distribution for each of their pure strategies.

Consider the game with the pay-off matrix $[a_{ij}]_{m \times n}$ such that the maximizing player A has m pure strategies and B has n pure strategies. Let the probabilities with which A and B select their pure strategies be given by

$$X = \{x_1, x_2, \ldots, x_i, \ldots, x_m\}, \quad x_i \geq 0 \ \forall i$$
$$\& \ Y = \{y_1, y_2, \ldots, y_j, \ldots, y_n\}, \quad y_j \geq 0 \ \forall j$$

which are called the mixed strategies for A and B respectively such that

$$x_1 + x_2 + \cdots + x_m = 1$$
$$\& \ y_1 + y_2 + \cdots + y_n = 1.$$

Now if a_{ij} represents the (i,j)th entry of the game then x_i and y_j will appear as follows.

$$\begin{array}{cc|cccc|}
 & & & B & & \\
 & & y_1 & y_2 & \cdots & y_n \\
\hline
 & x_1 & a_{11} & a_{12} & \cdots & a_{1n} \\
A & x_2 & a_{21} & a_{22} & \cdots & a_{2n} \\
 & \vdots & \vdots & \vdots & & \vdots \\
 & x_m & a_{m1} & a_{m2} & \cdots & a_{mn} \\
\hline
\end{array}$$

Here a_{ij} is the pay-off to player A when players A and B use the pure strategies A_i and B_j respectively. Then the expected pay-off to player A, given that player B uses his pure strategy B_j is

$$E(X, B_j) = \sum_{i=1}^{m} a_{ij}\, x_i$$

The expected pay-off to player A (called the payoff-function to A) when the player B uses the strategy Y is given by

$$E(X, Y) = \sum_{j=1}^{n} y_j\, E(X, B_j)$$

$$= \sum_{i=1}^{m} \sum_{j=1}^{n} a_{ij}\, x_i\, y_j$$

The mixed strategy problem is solved by applying the minimax criterion in which the maximizing player chooses the probabilities x_i which maximize the smallest expected gains in columns and the minimizing player chooses his probabilities y_j which minimize the largest expected gains in rows.

So that
$$\underline{v} = \max_{x_i} \left[\min_{y_j} \left\{ \sum_i a_{i1} x_i, \sum_i a_{i2} x_i, \cdots, \sum_i a_{in} x_i \right\} \right]$$

$$\bar{v} = \min_{y_j} \left[\max_{x_i} \left\{ \sum_j a_{1j} y_j, \sum_j a_{2j} y_j, \cdots \sum_j a_{mj} y_j \right\} \right]$$

These values are referred to as the maximin and the minimax expected payoff respectively.

Dominance Property :

We have discussed that the method of solving games with pure ~~strategy~~ strategies by maximin minimax principle. Further we have discussed the 2x2 game (having no saddle point) algebraically by using mixed strategies and the 2xn or mx2 games can be solved by graphically which is discussed later.

Here we discuss a method in which the pay-off matrix can be reduced by mere observations and in many cases it may be solved only by adopting this method. The method of reduction of the pay-off matrix by this process is called the dominance property of the rows and columns of the pay-off matrix.

General Rules for dominance :

We give below a summary of rules for dominance as deduced in the previous section.

(i) If all the elements of the ith row be less than or equal to the corresponding elements of any other row, say rth row, then rth row dominates the ith row and discard it.

(ii) If all elements in the jth column be greater than or equal to the corresponding elements of any column, say p-th, then the pth column is dominated by the jth column and we discard it.

In the case of row player the inferior row is discarded while in the case of column player the dominating column is discarded.

(iii) If the ith-~~player~~ row be dominated by the convex combination of the other rows, then the ith row is deleted from the pay-off matrix. If the jth column dominates a convex combination of other columns. then jth column is deleted from the pay-off-matrix.

(iv) R_1, R_2 are the ~~two~~ sub-sets of the rows of $m \times n$ pay-off matrix and C_1, C_2 are the sub-sets of its column. If a convex combination of rows (columns) in R_1 (C_1) dominates a convex combination of the rows (columns) in R_2 (C_2), then there exists a row (column) in R_2 (C_1) which is discarded.

by

Ex: Reduce the following game to 2×2 game by using dominance property and then solve the game.

$$A \quad \begin{array}{c} \\ A_1 \\ A_2 \\ A_3 \\ A_4 \end{array} \begin{array}{c} \quad\quad\quad B \\ \begin{array}{cccc} B_1 & B_2 & B_3 & B_4 \end{array} \\ \begin{bmatrix} 1 & 2 & -2 & 2 \\ 3 & 1 & 2 & 3 \\ -1 & 3 & 2 & 1 \\ -2 & 2 & 0 & -3 \end{bmatrix} \end{array}$$

al

Step 1: All the elements of the 4th ~~entry~~ row are less than the corresponding elements of 3rd row. Thus R_4 is dominated by R_3 and thus drop the fourth row. The resulting matrix is given below

$$\begin{array}{c} \\ A_1 \\ A_2 \\ A_3 \end{array} \begin{array}{c} \begin{array}{cccc} B_1 & B_2 & B_3 & B_4 \end{array} \\ \begin{bmatrix} 1 & 2 & -2 & 2 \\ 3 & 1 & 2 & 3 \\ -1 & 3 & 2 & 1 \end{bmatrix} \end{array}$$

Step 2: All the elements of 4th column are greater or equal to all the corresponding elements of 1st column. Then the resulting matrix is given below

$$\begin{array}{c|ccc} & B_1 & B_2 & B_3 \\ \hline A_1 & 1 & 2 & -2 \\ A_2 & 3 & 1 & 2 \\ A_3 & -1 & 3 & 2 \end{array}$$

Step 3: By using the convex combination of 2nd row and 3rd row we have $\frac{1}{2}R_2 + \frac{1}{2}R_3 = [1, 2, 2]$, all the elements are greater or equal to the corresponding elements of the 1st row. Then drop the 1st row and the resulting matrix is given below

$$\begin{array}{c|ccc} & B_1 & B_2 & B_3 \\ \hline A_2 & 3 & 1 & 2 \\ A_3 & -1 & 3 & 2 \end{array}$$

Step 4: Now one convex combination of C_1 and C_2 is $\frac{1}{2}C_1 + \frac{1}{2}C_2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ are less than or equal to the corresponding elements of the 3rd column and thus drop the 3rd column and the resulting matrix is given below

$$\begin{array}{c|cc} & B_1 & B_2 \\ \hline A_2 & 3 & 1 \\ A_3 & -1 & 3 \end{array}$$

Now using the formula for solving 2×2 game. We have

$$p_2^* = \frac{3-(-1)}{3+3-(1-1)} = 2/3$$

$$p_3^* = 1 - 2/3 = 1/3$$

$$q_1^* = \frac{3-1}{3+3-(1-1)} = \frac{1}{3}.$$

$$q_2^* = 1 - 1/3 = 2/3$$

And the value of the game is $\dfrac{3\times 3 - (1\times -1)}{3+3-(1-1)} = 5/3$

Thus the optimal strategies are

$$p^* = \left(0, \tfrac{2}{3}, \tfrac{1}{3}, 0\right), \quad q^* = \left(\tfrac{1}{3}, \tfrac{2}{3}, 0, 0\right),$$

$$v = 5/3.$$

Ex-2: Solve the game problem by reducing 2 × 2 problem using the dominance property.

$$\begin{array}{cc} & B \\ A & \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 4 & 2 & 0 & 2 & 1 \\ 4 & 3 & 1 & 3 & 2 \\ 4 & 3 & 4 & -1 & 2 \end{bmatrix} \end{array}$$

Ans: We can simply drop the 1st and 2nd row as they are dominated by 3rd row. Similarly, the 1st and 2nd column can be dropped as they are dominated by 4th and 5th column. The resultant matrix is

$$\begin{array}{cc} & \begin{array}{ccc} B_3 & B_4 & B_5 \end{array} \\ \begin{array}{c} A_3 \\ A_4 \end{array} & \begin{bmatrix} 1 & 3 & 2 \\ 4 & -1 & 2 \end{bmatrix} \end{array}$$

Now one convex combination of C_3 & C_4 give $\frac{1}{2}(C_3 + C_4) = (2, 3/2)$. Then C_5 is dominated by the convex combination of C_3 & C_4. Hence drop C_5 and the resultant matrix is

$$\begin{array}{cc} & \begin{array}{cc} B_3 & B_4 \end{array} \\ \begin{array}{c} A_3 \\ A_4 \end{array} & \begin{bmatrix} 1 & 3 \\ 4 & -1 \end{bmatrix} \end{array}$$

Now solving algebracilly, we have

$$p_3^* = \frac{-1-4}{1-1-(4+3)} = 5/7, \quad p_4^* = 1 - 5/7 = 2/7$$

$$q_3^* = \frac{-1-3}{1-1-(4+3)} = \frac{4}{7}, \quad q_4^* = 1-4/7 = 3/7.$$

and the value of the game $= \frac{13}{7}$

Hence the optimal strategies are

$$p^* = \left(0, 0, \frac{5}{7}, \frac{2}{7}\right), \quad q^* = \left(0, 0, \frac{4}{7}, \frac{3}{7}, 0\right)$$

$$v = \frac{13}{7}.$$

Graphical solution of $2\times n$ or $m\times 2$ games:

We have discussed the method of finding out the value of the game of 2×2 pay-off matrix (without any saddle point), algebrically. But it is not possible to solve easily any rectangular game of $m\times n$ order algebrically. But using graph, it is possible to reduce any rectangular game of order $2\times n$ or $m\times 2$ to a 2×2 game and then it can be solved by algebric method.

Let a particular $2\times n$ game be

$$\begin{array}{cc|c} & & B \\ & & \begin{array}{cccc} B_1 & B_2 & \cdots & B_n \end{array} \\ A & \begin{array}{c} A_1 \\ \\ A_2 \end{array} & \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \\ a_{21} & a_{22} & \cdots & a_{2n} \end{bmatrix} \end{array}$$

without any saddle point. Let the mixed strategies used by A be $p = (p_1, p_2)$ and B be $q = (q_1, q_2, \cdots, q_n)$. Then the net expected gain of A when B plays his pure strategies B_j is given by

$$E_j(p) = a_{1j}p_1 + a_{2j}p_2, \quad j = 1, 2, \cdots, n,$$

where $p_1 + p_2 = 1$. and p_1 & p_2 lies in $(0,1)$.

Here $E_j(p)$ is a linear function of either p_1 or p_2.

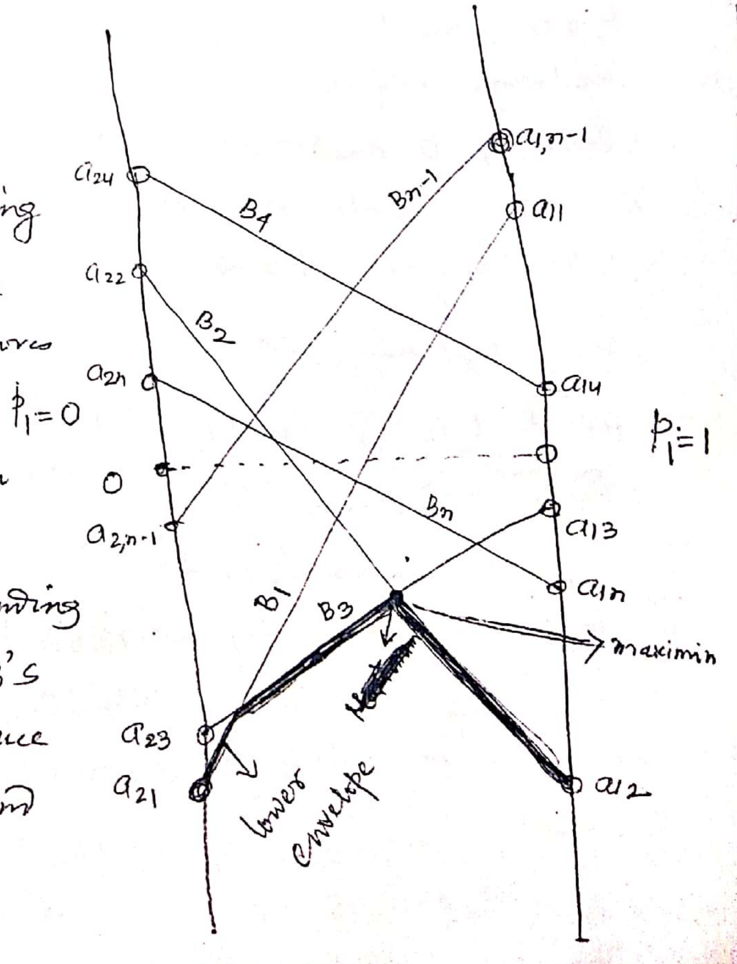
Considering $E_j(p)$ is a linear function of p_1.

$$\therefore\ E_j(p) = a_{2j} \quad \text{for } p_1 = 0$$
$$= a_{1j} \quad \text{for } p_1 = 1.$$

Hence $E_j(p)$ represents line segment joining the points $(0, a_{2j})$ & $(1, a_{1j})$.

Let us draw two parallel vertical lines, distance between them being one unit length, first one represents the line $p_1 = 0$ and the 2nd one represents $p_1 = 1$. Now draw n line segments joining the points $(0, a_{2j})$ and $(1, a_{1j})$, for $j = 1, 2, \ldots, n$. The lower envelope of these line segments (indicate it by thick line segments) will give minimum expected gain of A as a function of p_1. Now the highest point of the lower envelope will give the maximum of minimum gain of A.

The line segments passing through the point corresponding to B's two pure moves say B_K & B_L are the critical moves for B which will maximize the minimum expected gain of A. Now the 2x2 pay-off matrix corresponding to A's moves A_1 & A_2 and B's moves B_K & B_L will produce the required results. and



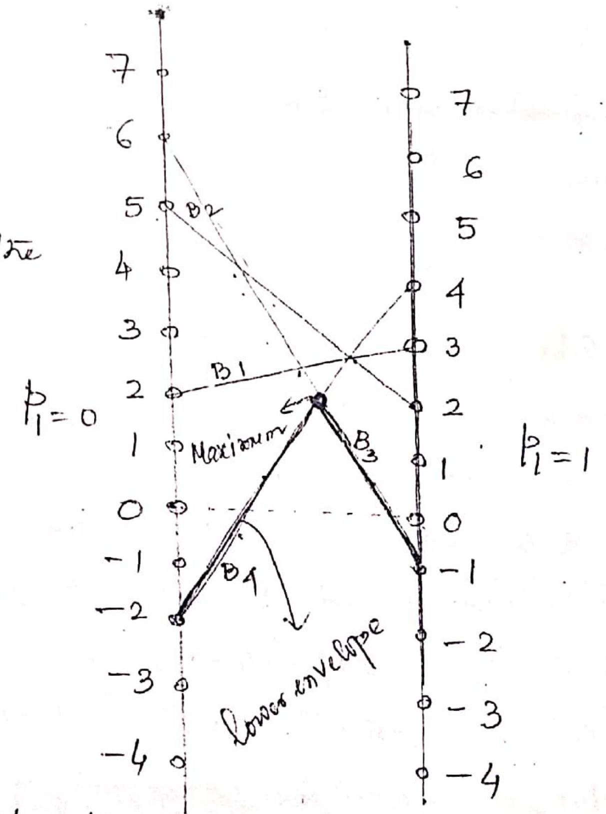
Thus solving the 2×2 pay-off-matrix algebrically we shall be in a position to determine the value of the game. Using the similar technique we may solve any m×2 game.

(EX) Solve the following 2×4 game graphically

$$
\begin{array}{cc}
 & \begin{array}{cccc} & B & & \\ B_1 & B_2 & B_3 & B_4 \end{array} \\
A \begin{array}{c} A_1 \\ A_2 \end{array} & \begin{bmatrix} 3 & 2 & -1 & 4 \\ 2 & 5 & 6 & -2 \end{bmatrix}
\end{array}
$$

Ans: The problem does not possess a saddle point. Let the player A play with mixed strategy $p = (p_1, p_2)$, $p_1 + p_2 = 1$ and both p_1, p_2 lies in the open interval $(0,1)$

Draw two vertical lines $p_1 = 0$ and $p_1 = 1$ at unit distance apart. Mark the lines $p_1 = 0$ and $p_1 = 1$ using the same scale as given in the figure. Draw the line segment joining the point $(0,2)$ and $(1,3)$ which represents A's expected gain due to B's pure move B_1. Number the line segment as B_1. Similarly draw line segments joining the points $(0,5)$ and $(1,2)$ which represents A's ~~exped~~



expected gain due to B's pure move B_2. Number the line segment as B_2. Similarly ~~draw~~ draw the line segment B_3 & B_4. The maximin point of the lower envelope lies on the point of intersection of the line segments B_3 & B_4.

Therefore, the game can be solved ultimately solving the 2x2 pay-off matrix given below

$$\begin{array}{ccc} & & B \\ & B_3 & B_4 \\ A \quad A_1 & -1 & 4 \\ \quad\;\; A_2 & 6 & -2 \end{array}$$

with mixed strategies $p = (p_1, p_2)$, $p_1 + p_2 = 1$ for A

and $q = (q_3, q_4)$, $q_3 + q_4 = 1$ for B.

using the formula,

$$p_1^* = \frac{-2-6}{-1-2-(6+4)} = \frac{8}{13}$$

$$p_2^* = 1 - \frac{8}{13} = \frac{5}{13}.$$

$$\& \; q_3^* = \frac{-2-\cancel{4}}{-1-2-(6+4)} = \frac{6}{13} \; \& \; q_4^* = 1 - \frac{6}{13} = 7/13$$

& the value of the game

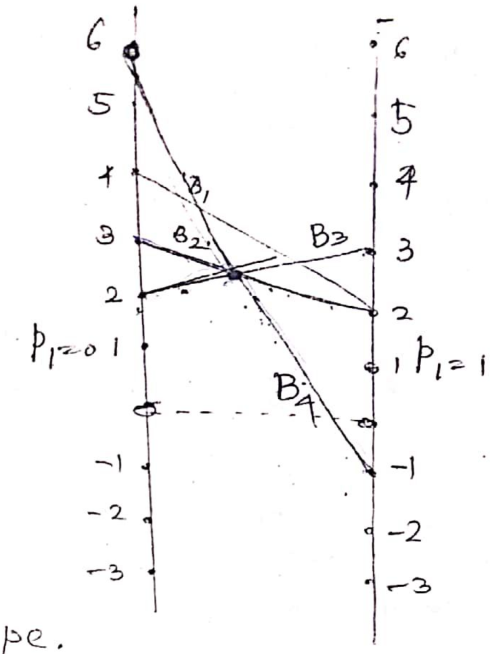
$$= \frac{(-1)\times(-2) - 6\times 4}{-1-2-(6+4)} = \frac{22}{13} = v.$$

Hence ultimately the solution of the problem is given by $p^* = \left(\frac{8}{13}, \frac{5}{13}\right)$ for A & $q^* = \left(0, 0, \frac{6}{13}, \frac{7}{13}\right)$ for B and the value of the game is 22/13.

Ex-2: Solve the following 2×4 game by graphical method.

$$\begin{array}{cc} & \begin{array}{cccc} B_1 & B_2 & B_3 & B_4 \end{array} \\ A \begin{array}{c} A_1 \\ A_2 \end{array} & \begin{bmatrix} 2 & 2 & 3 & -1 \\ 4 & 3 & 2 & 6 \end{bmatrix} \end{array}$$

Ans: Here the three lines passes at the highest point of the lower envelope. Thus accordingly we get $3_{c_2} = 3$ square matrices of order 2 and there are three optimal solutions. But actually, we shall have to select such pairs of lines which have opposite sign ~~envelope~~ slope.



Thus one square matrix is (B_2, B_3) & other is (B_3, B_4) and the square matrices are

$$\begin{array}{cc} & \begin{array}{cc} B_2 & B_3 \end{array} \\ \begin{array}{c} A_1 \\ A_2 \end{array} & \begin{bmatrix} 2 & 3 \\ 3 & 2 \end{bmatrix} \end{array} \quad \text{and} \quad \begin{array}{cc} & \begin{array}{cc} B_3 & B_4 \end{array} \\ \begin{array}{c} A_1 \\ A_2 \end{array} & \begin{bmatrix} 3 & -1 \\ 2 & 6 \end{bmatrix} \end{array}$$

and solving by usual method, the value of the game is $5/2$ and the optimal solutions are

$p^* = (1/2, 1/2)$, $q^* = (0, 1/2, 1/2, 0)$ and

$p^* = (1/2, 1/2)$, $q^* = (0, 0, 7/8, 1/8)$.

Ex-3: Solve the 3×2 game graphically

$$\begin{array}{cc} & \begin{array}{cc} B_1 & B_2 \end{array} \\ A \begin{array}{c} A_1 \\ A_2 \\ A_3 \end{array} & \begin{bmatrix} 2 & -3 \\ -2 & 5 \\ 0 & -1 \end{bmatrix} \end{array}$$

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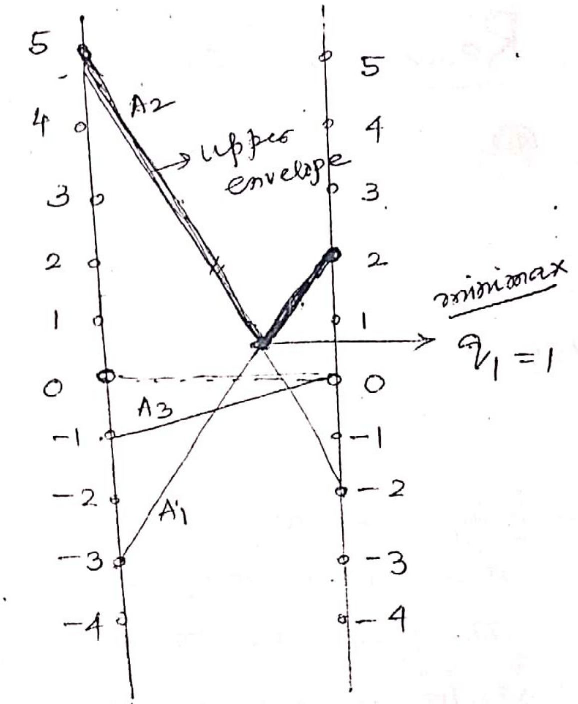
Ans: This problem does not possess a saddle point.

To solve the problem, draw two vertical straight line at a unit distance apart, the equation of the 1st being $q_1 = 0$ & the 2nd being $q_1 = 1$. Join $(0, -3)$ and $(1, 2)$ to get the line segment corresponding to A's pure move A_1. Similarly join $(0, 5)$ & $(1, -2)$ to get the line segment corresponding to A's pure move A_2.

From the figure we get the minimax point of the upper envelope, which is the point of intersection of A's two critical moves A_1 and A_2.



Therefore the problem ultimately can be solved by solving 2×2 pay-off matrix

$$\begin{array}{c|cc} & B_1 & B_2 \\ A_1 & 2 & -3 \\ A_2 & -2 & 5 \end{array}$$

with mixed strategies $p = (p_1, p_2)$, $p_1 + p_2 = 1$ for A and $q = (q_1, q_2)$, $q_1 + q_2 = 1$ for B.

Now, using the formula,

$$p_1^* = \frac{5-(-2)}{2+5-(-2-3)} = 7/12, \quad p_2^* = 1 - 7/12 = 5/12, \quad p_3^* = 0$$

$$q_1^* = \frac{5-(-3)}{2+5-(-2-3)} = 2/3, \quad q_2^* = 1 - 2/3 = 1/3$$

and the value of the game is

$$v = \frac{2\times5-(-2)\times(-3)}{2+5-(-2-3)} = \frac{4}{12} = \frac{1}{3}$$

Fundamental theorem of rectangular game:

If mixed strategies be allowed, the value of a game exists and unique (optimal strategies may be different).

Reduction of a game problem to a L.P.P

Every game problem (two person zero-sum game) can be converted a L.P.P.

Ans: Let the payoff matrix of a game problem be a_{ij}, $i=1(1)m$ and $j=1(1)n$ and with proper adjustment [by adding a suitable constant to each element of the matrix] we may assume $a_{ij} > 0$, $\forall i, j$. i.e the value of the game exists and then it will be a positive quantity.

Let the mixed strategies used by A and B be

$$p = (p_1, p_2, \dots, p_m), \quad \sum_{i=1}^{m} p_i = 1 \quad p_i \geq 0 \ \forall i$$

$$q = (q_1, q_2, \dots, q_n), \quad \sum_{j=1}^{n} q_j = 1, \ q_j \geq 0 \quad \forall j \text{ respectively.}$$

The net expected gain of A when B plays his pure move B_j is

$$E_j(p) = \sum_{i=1}^{m} a_{ij} p_i, \quad j = 1, 2, \dots, n.$$

Now A will expect a least possible gain u. Hence

$$E_j(p) = \sum_{i=1}^{m} a_{ij} p_i \geq u, \quad j = 1, 2, \dots, n$$

A

As all $a_{ij} > 0$, then u will essentially a positive quantity.

Now A's problem is to select $p_1, p_2, \dots, p_m$ in such a way that they will satisfy the conditions stated above and u attains its maximum.

Since $u > 0$, maximization of u is equivalent to minimization of $\frac{1}{u}$. Now dividing $E_j(P) \geq u$ by u, A's problem can be written as

$$\text{Minimize} \left\{ p_0 = \frac{1}{u} = \frac{p_1 + p_2 + \dots + p_m}{u} = p_1' + p_2' + \dots + p_m' \right\}$$

$$\text{s.t.} \quad \sum_{i=1}^{m} a_{ij} p_i' \geq 1, \quad j = 1, 2, \dots, n \quad \text{where}$$

$$p_i' = \frac{p_i}{u} \geq 0, \quad i = 1, 2, \dots, m$$

Using the same logic, B's problem can be written as

$$\text{Maximize} \left\{ q_0 = \frac{1}{u} = \frac{q_1 + q_2 + \dots + q_n}{u} = q_1' + q_2' + \dots + q_n' \right\}$$

$u > 0$

$$\text{s.t.} \quad \sum_{j=1}^{n} a_{ij} q_j' \leq 1, \quad i = 1, 2, \dots, m \quad \text{where}$$

$$q_j' = \frac{q_j}{u} \geq 0 \quad j = 1, 2, \dots, n.$$

Thus considering from A and B's point of view, we find that a game problem can be reduced to a L.P.P.

(Ex)- Solve the game problem by converting it into a L.P.P

$$\begin{array}{c|c} & \begin{array}{cc} B_1 & B_2 \end{array} \\ \begin{array}{c} A_1 \\ A_2 \end{array} & \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix} \end{array}$$

Ans : The value of the game may not be positive. Adding 2 to each element, we get a pay-off matrix whose value will be essentially positive and solving 2nd problem we can find the value of the original problem. The pay-off-matrix after such addition is $\begin{bmatrix} 5 & 3 \\ 1 & 4 \end{bmatrix}$

Let the optimal strategies for A and B be $p^* = (p_1^*, p_2^*)$ & $q^* = (q_1^*, q_2^*)$.

Now considering the B's problem it can be reduced to

Maximize $q_0 = q_1' + q_2'$ subject to

$$5q_1' + 3q_2' \leq 1.$$

$$q_1' + 4q_2' \leq 1, \quad q_1' \geq 0, q_2' \geq 0.$$

After using the simplex method

		c	1	1	0	0	
Basic	C_B	b	a_1	a_2	a_3	a_4	Min
a_3	0	1	$\boxed{5}$	3	1	0	$1/5 \rightarrow$
a_4	0	1	1	4	0	1	1
$z_j - c_j$		0	$-1\uparrow$	-1	0	0	
a_1	1	$1/5$	1	$3/5$	$1/5$	0	$1/3$
a_4	0	$4/5$	0	$\boxed{17/5}$	$-1/5$	1	$\frac{4}{17} \rightarrow$
$z_j - c_j$		$1/5$	0	$-2/5\uparrow$	$1/5$	0	
a_1	1	$\frac{1}{17}$	1	0	$4/17$	$-3/17$	
a_2	1	$\frac{4}{17}$	0	1	$-\frac{1}{17}$	$\frac{5}{17}$	
$z_j - c_j$		$5/17$	0	0	$3/17$	$2/17$	

Then Max $q_0 = \frac{5}{17} = \frac{1}{v^*}$ (say)

at $q_1'^* = \frac{1}{17}$, $q_2'^* = \frac{4}{17}$

Then the value of the original game is

$v^* - 2 = \frac{17}{5} - 2 = 7/5$

at $q_1^* = q_1'^* \times v^* = \frac{1}{17} \times \frac{17}{5} = \frac{1}{5}$

& $q_2^* = q_2'^* \times v^* = \frac{4}{17} \times \frac{17}{5} = \frac{4}{5}$

Now using the duality theory, we have

$p_1'^* = \frac{3}{17}$, $p_2'^* = \frac{2}{17}$

$\therefore p_1^* = p_1'^* \times v^* = \frac{3}{17} \times \frac{17}{5} = 3/5$

$p_2^* = p_2'^* \times v^* = \frac{2}{17} \times \frac{17}{5} = 2/5$.

Thus the optimal strategies are

$$p^* = (3/5, 2/5), \ q^* = (1/5, 4/5)$$

$$\& \ v = 7/5.$$

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